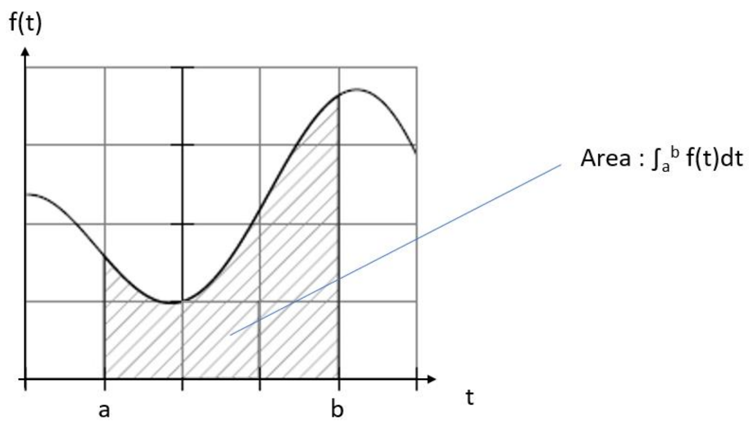
Integration and Primitives Essentials

1. Introduction

Integrals together with derivatives are fundamental objects in Calculus; a very clear conceptual understanding of these is a must. This chapter summarizes the principles of integration and the link between integrals and primitives.

2. DEFINITE INTEGRAL - DEFINITION

Let f denote a continuous and positive function on an interval [a, b].
By definition the Definite Integral of f between a and b, denoted by $\int_a^b f$ or $\int_a^b f(t)dt$, is the area between the f curve and the abscissa axis, delimited by a and b.



It is called a "definite" integral because of its dependence on the two given constants a and b.

The concept can be extended to a non-positive function, bearing in mind that areas in the negative portions of the function are negative.

3. INDEFINITE INTEGRAL or PRIMITIVE

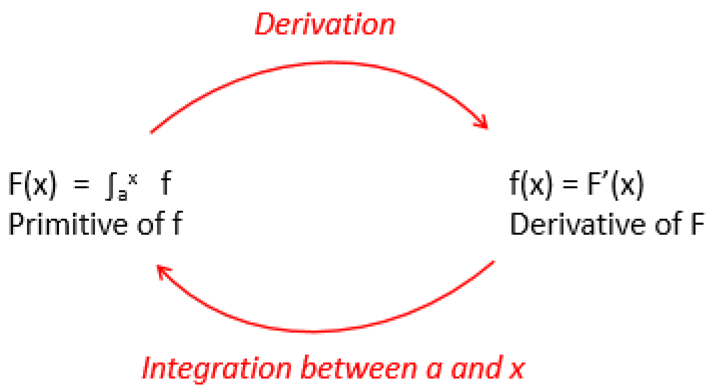
The Indefinite Integral or Primitive is a generalization of the Definite Integral. It is a function (as opposed to a definite value) depending on a variable, say x, which replaces the constant value b.

Notation: $F(x) = \int_a^x f$ also written as $\int_a^x f(t)dt$

KEY PROPERTIES OF PRIMITIVES

- F'(x) = f(x): given that $F(x) = \int_a^x f$, then the derivative of F is f ; the primitive can be looked at as the "inverse" of the derivative.
- If F is a primitive of f then F plus any constant is also a primitive of f, since the derivative of a constant is 0; so there is an infinite number of primitives of a given function f all differing by a constant term.
- $\int_a^b f = F(b) - F(a)$: formula to calculate a Definite Integral as the difference of the primitive at two given points b and a.

IN SUMMARY



4. EXAMPLE APPLICATIONS

1- Direct calculation of $\int_a^b f$ for a given function f

$\int_a^b f = F(b) - F(a)$ where F is the primitive of f .

It is then just a matter of identifying F given the function f, based on the knowledge of derivatives of common functions provided one of them is applicable.

Simple example:

- f(x) = x ; find $\int_a^b f = \int_a^b x$
- Derivative of x^2 is 2x, therefore derivative of $x^2/2$ is x
- Primitive de f(x) : $F(x) = x^2/2$
- $\int_a^b f = b^2/2 - a^2/2$

2- Integration by parts: a useful technique for finding the integral of a function when expressed as a product uv' where v' is a derivative of which we know the primitive.

Formula: $\int uv' = uv - \int u'v$

Example 1: ∫ x cos(x)

- u = x   v' = cos(x)
- u' = 1   v = sin(x)
- ⇒ ∫ x cos(x) = x sin(x) - ∫ sin(x) = x sin(x) + cos(x) + constant

Example 2: ∫ x ln(x)

- u = ln(x)   v' = x
- u' = 1/x   v = $x^2/2$
- ⇒ ∫ x ln(x) = x^2 ln(x)/2 - ∫(1/x) ($x^2/2$) = x^2 ln(x)/2 - ∫ x/2 = x^2 ln(x)/2 - $x^2/4$ + constant

3- Integration by substitution: a useful technique for finding the integral of a function when expressed as a product of a composite function g ∘ f (x) = g [f(x)] and of the derivative of f.

Formulae: ∫ g [f(x)] f'(x) dx = ∫ g(y) dy
after substituting f(x) for y and f'(x) dx for dy.

Example: ∫ sin(√x) / √x dx

- y = √x
- dy = 1 / 2√x dx
- ⇒ ∫ sin(√x) / √x dx = ∫ sin(y) * (2 dy) = 2 ∫ sin(y) dy = -2 cos(y) = -2 cos(√x) + constant