A close look at quadratic equations

1. Introduction

A rather common topic in maths: how to best solve quadratic equations. Not really rocket science, but there are a few tricks to do it fast, systematically and minimize chances of errors.

CHOOSE THE SIMPLE METHODS FIRST TO SAVE TIME, DEPENDING ON THE PROBLEM AT HAND !

2. Sequential approach

Let us take the usual formula for a quadratic equation: $f(x) = ax^2 + bx + c = 0$.

1) First, if you can recognize one of the remarkable identities, for instance $\alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta)$, then you are immediately done. For the $\alpha^2 - \beta^2$ situation, there are two roots corresponding to $\alpha = \beta$ and $\alpha = -\beta$.

   Example: $f(x) = 4x^2 - 4x + 1$
   This is the same pattern as the identity $\alpha^2 - 2\alpha\beta + \beta^2 = (\alpha - \beta)^2$
   Therefore: $f(x) = (2x - 1)^2$ with a double root $x = 1/2$

2) When case 1 is not applicable, maybe it is possible to find an obvious root of the equation by direct observation. This often happens, for instance roots such as 1, 2, -1 ,… Let us call this obvious root x_1. Since x_1 is a root then you know that f(x) needs to be of the form $(x - x_1)(ax + d)$ and the problem is almost solved. The constant terms in $ax^2 + bx + c$ and $(x - x_1)(ax + d)$ have to match, therefore $c = -d\,x_1$ and $d = -c / x_1$. As a consequence the second root is $x_2 = -d/a = -c/(ax_1)$

   Example: $g(x) = 2x^2 + 5x - 7 = 0$
   One obvious solution $x = 1$
   $g(x) = (x - 1)(2x + \text{constant})$; the constant can be easily seen to be equal to 7.
   Therefore: $g(x) = (x - 1)(2x + 7)$ with roots 1 and -7/2

3) If the quick first and second methods are not applicable, the traditional “quadratic formula” method including the calculation of the discriminant $b^2 - 4ac$ can be applied; it works all the time, however at the expense of the required calculations.

   An interesting and effective alternative to the quadratic formula is to transform f(x) into its so-called “canonical form” $a(x - \alpha)^2 + \beta$. The method is also called “solving by completing the square”:

   - For the x-terms to match between $ax^2 + bx + c$ and $a(x - \alpha)^2 + \beta = a(x^2 - 2\alpha x + \alpha^2) + \beta$, you need $b = -2\alpha a$, therefore $\boxed{\alpha = -b/2a}$.

- Since ax^2 is replaced by $a(x-\alpha)^2$ in the expression $ax^2 + bx + c$, in order to get the constant terms correct, we need to subtract $a\alpha^2$:

  $f(x) = ax^2 + bx + c = a(x-\alpha)^2 - a\alpha^2 + c$; this gives $\boxed{\beta = -a\alpha^2 + c}$.

- Conclusion: the canonical form of $ax^2 + bx + c$ is $a(x-\alpha)^2 + \beta$, with $\alpha = -b/2a$ and $\beta = -a\alpha^2 + c$
  .
- The roots can be derived directly from the canonical form by solving $a(x-\alpha)^2 + \beta = 0$, which has real roots only if $\beta/a \leq 0$.

Example: $h(x) = x^2 - 4x - 1$

$h(x)$ can be re-written as $h(x) = (x^2 - 4x + 4) - 4 - 1 = (x-2)^2 - 5$

The roots are such that $(x-2)^2 = 5$ i.e.

$x - 2 = \pm\sqrt{5}$

therefore the two roots are:

$2 + \sqrt{5}$; $2 - \sqrt{5}$

As a bonus, the canonical form gives directly the axis of symmetry ($x = \alpha$) of the parabolic curve corresponding to $f(x) = ax^2 + bx + c$ as well as the coordinates of the extremum (α, β) of the parabol.